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Reg. No. : ....

Code No. : 6381 **Sub. Code : ZMAE 31**

M.Sc. (CBCS) DEGREE EXAMINATION, NOVEMBER 2022.

Third Semester

Mathematics

Elective — ALGEBRAIC NUMBER THEORY

(For those who joined in July 2021 onwards)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. Which of the following Diophantine equation cannot be solved?
   (a) $6x+51y=22$ (b) $33x+14y=115$
   (c) $14x-35y=93$ (d) $11x+13y=21$

2. The linear Diophantine equation $ax+by=c$ has a solution if and only if ———
   (a) $\gcd(a,c)|b$ (b) $\gcd(a,b)|c$
   (c) $\gcd(c,b)|a$ (d) $c|\gcd(a,b)$

3. A linear combination of integers a and b is
   (a) ab
   (b) $\frac{a}{x}+\frac{b}{y}$, x and y are integers
   (c) $ab=1$
   (d) $ax+by$, x and y are integers

4. Let a and b be integers, not both zero. Then a and b are relatively prime iff there exists integers x and y such that
   (a) $1=ax+by$ (b) $2=ax+by$
   (c) $ab=ax+by$ (d) $a-b=ax+by$

5. The number of prime is
   (a) finite (b) infinite
   (c) uncountable (d) 1729

6. Two integers a and b, not both of which are zero, are said to be relatively prime if
   (a) $\gcd(a,b)=a$ (b) $a|b$
   (c) $\gcd(a,b)=1$ (d) $b|a$

7. If a is a solution of $P(x) \equiv 0 (\bmod n)$ and $a \equiv b (\bmod n)$, then

(a) ab is also a solution

(b) $a+b$ is also a solution

(c) $a-b$ is also a solution

(d) b is also a solution

8. In n is an odd pseudo prime, then $2^n - 1$ is

(a) pseudo prime (b) prime

(c) irrational (d) not pseudo prime

9. If m and n are relatively prime integers then $\varphi(mn) =$ ________

(a) $\varphi(m) + \varphi(n)$ (b) $\varphi(m) / \varphi(n)$

(c) $\varphi(m) - \varphi(n)$ (d) $\varphi(m)\varphi(n)$

10. If p is a prime and a is any integer then $a^p - a$ is

(a) a multiple of p^2 (b) a multiple of $p-1$

(c) a multiple of $2p$ (d) a multiple of p

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).
Each answer should not exceed 250 words.

11. (a) Find all solutions in positive integer $5x + 3y = 52$.

Or

(b) If u and v are relatively prime positive integers whose product uv is a perfect square, then prove that u and v are both perfect squares.

12. (a) Prove that the equation $x + 2y^3 + 4z^3 = 9w^3$ has no nontrivial solution.

Or

(b) Determine the solution of the Diophantine equation $x^2 + 3y^2 + 5z^2 + 7xy + 9yz + 11zx = 0$.

13. (a) For any positive real number x, prove that $\langle a_0, a_1, \ldots, a_{n-1}, x \rangle = \dfrac{xh_{n-1} + h_{n-2}}{xk_{n-1} + k_{n-2}}$.

Or

(b) Let $\theta = \langle a_0, a_1, a_2, \ldots \rangle$ be a simple continued fraction. Then $a_0 = [\theta]$. Further more if θ_1 denotes $\langle a_0, a_1, a_2, \ldots \rangle$ then prove that $\theta = a_0 + 1/\theta_1$.

14. (a) Let ξ denote any irrational number. If there is a rational number $\frac{a}{b}$ with $b \geq 1$ such that $\left|\xi - \frac{a}{b}\right| < \frac{1}{2b^2}$ then prove that $\frac{a}{b}$ equals one of the convergents of the simple continued fraction expansion of ξ.

Or

(b) If an irreducible polynomial $p(x)$ divided a product $f(x)g(x)$, then prove that $p(x)$ divides at least one of the polynomials $f(x)$ and $g(x)$.

15. (a) Let m be a negative square-free rational integer Prove : the field $\mathbb{Q}(\sqrt{m})$ has units ± 1, and these are the only units except in the cases $m = -1$ and $m = -3$. The units for $\mathbb{Q}(i)$ are ± 1 and $\pm i$. The units for $\mathbb{Q}\sqrt{-3}$ are ± 1, $(1 \pm \sqrt{-3})/2$ and $\left(-1 \pm \sqrt{-3}/2\right)$

Or

(b) Prove that the integers of any algebraic number field form a ring.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b)
Each answer should not exceed 600 words.

16. (a) Find all solutions in integers of the simultaneous equations $20x + 44y + 50z = 10$, $17x + 13y + 11z = 19$.

Or

(b) Find all integers x and y such that $147x + 258y = 369$.

17. (a) Determine whether the equation $3x^2 + 5y^2 + 7z^2 + 9xy + 11yz + 13zx = 0$.

Or

(b) Prove that the equation $y^2 = x^3 + 7$ has no solution in integers.

18. (a) Prove that the equations $h_i k_{i-1} - h_{i-1} k_i = (-1)^{i-1}$ and $r_i - r_{i-1} = \frac{(-1)^{i-1}}{k_i k_{i-1}}$ hold for $i \geq 1$.

Or

(b) Prove that the value of any infinite simple continued fraction $\langle a_0, a_1, a_2, \ldots \rangle$ is irrational.

19. (a) If a/b is a rational number with positive denominator such that $\left|\xi - \frac{a}{b}\right| < \left|\xi - \frac{h_n}{k_n}\right|$ for some $n \geq 1$, then $b > k_n$. In fact if $|\xi b - a| < |\xi k_n - h_n|$ for some $n \geq 0$, then prove that $b \geq k_{n+1}$.

Or

(b) Prove that any periodic simple continued fraction is a quadratic irrational number and conversely.

20. (a) The norm of a product equals the product equals the product of the norms, $N(\alpha\beta) = N(\alpha)N(\beta)$. $N(\alpha) = 0$ iff $\alpha = 0$. The norm of an integer in $\mathbb{Q}(\sqrt{m})$ is a rational integer. If γ is an integer in $\mathbb{Q}(\sqrt{m})$, then prove that $N(\gamma) = \pm 1$ iff γ is a unit.

Or

(b) Prove that the fields $\mathbb{Q}(\sqrt{m})$ for $m = -1, -2, -3, -7, 2, 3$, are Euclidean and so have the unique factorization property.